

Elliptic fibrations on K3 surfaces

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Abstract

The present paper consists mainly of a review and applications of our old results related to the title. We discuss how many elliptic fibrations and elliptic fibrations with infinite automorphism groups (or Mordell–Weil groups) an algebraic K3 surface over an algebraically closed field can have.

As examples of applications of the same ideas, we also consider K3 surfaces with exotic structures: with finite number of non-singular rational curves, with finite number of Enriques involutions, and with naturally arithmetic automorphism groups.

Dedicated to my old friend and colleague Slava Shokurov
on occasion of his 60th birthday

1 Introduction

This is mainly a review and applications of our old results related to elliptic fibrations on K3 surfaces over algebraically closed fields. See [4]—[11]. The most important are our papers [5], [6], [10] and [11].

This was the subject of our talk at the Oberwolfach workshop “Higher dimensional elliptic fibrations” in October 2010. Elliptic fibrations are especially interesting for Fano and Calabi–Yau varieties. Thus, it is interesting to study these fibrations in the case of K3 surfaces which are 2-dimensional Calabi–Yau manifolds.

We consider algebraic K3 surfaces X over arbitrary algebraically closed fields k .

In Section 2, we discuss basic results by Piatetsky-Shapiro and Shafarevich [12]. In particular, we discuss, when a K3 surface X has an elliptic fibration.

In Section 3, we discuss, when a K3 surface X has an elliptic fibration with infinite automorphism group (or the Mordell–Weil group). See [5].

In Section 4, we discuss our general results from [5], [10] and [11] on existence of non-zero exceptional elements of the Picard lattice with respect to the automorphism group of a K3 surface. Here an element x of the Picard lattice S_X is called *exceptional* with respect to the automorphism group $\text{Aut } X$, if its orbit $\text{Aut } X(x)$ in S_X is finite. These results will give the main tools for further applications.

In Section 5 (see also Section 4), we discuss, how many elliptic fibrations and elliptic fibrations with infinite automorphism group a K3 surface can have. In particular, for the Picard number $\rho(X) \geq 3$, we show that a K3 surface X has infinite number of elliptic fibrations and infinite number of elliptic fibrations with infinite automorphism groups if it has one of them, except a finite number of exceptional Picard lattices S_X . This is mainly related to our results in [5], [6], [10] and [11].

As examples of applications of the same ideas, in Section 6, we consider K3 surfaces with exotic structures: finite number of non-singular rational curves, finite number of Enriques involutions, and with naturally arithmetic automorphism groups.

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2 Results by Piatetsky-Shapiro and Shafarevich about existence of elliptic fibrations on K3 surfaces

We remind that an algebraic K3 surface X is a non-singular projective algebraic surface over an algebraically closed field k such that the canonical class $K_X = 0$ and the irregularity $q(X) = \dim H^1(X, \mathcal{O}_X) = 0$.

Further in this section, X is an algebraic K3 surface over an algebraically closed field. We denote by S_X the Picard lattice of X . It is well-known that S_X is a hyperbolic (i.e., of signature $(1, \rho(X) - 1)$) even integral lattice of rank $\rho(X)$ where $\rho(X) = \text{rk } S_X$ is the Picard number of X . It can be a very arbitrary even hyperbolic lattice of rank $\rho(X) \leq 22$, and it is an important invariant of X . It will be the most important for us.

According to Piatetsky-Shapiro and Shafarevich [12], elliptic fibrations

on X are in one to one correspondence with primitive isotropic numerically effective (i.e., *nef*) elements $c \in S_X$. That is $c \neq 0$, $c^2 = 0$; $c/n \in S_X$ only for integers $n = \pm 1$; $c \cdot D \geq 0$ for any effective divisor D on X . For such $c \in S_X$, the complete linear system $|c|$ is one-dimensional without base points, and it gives an elliptic fibration $|c| : X \rightarrow \mathbb{P}^1$, that is the general fibre is an elliptic curve (for char $k = 2$ or 3 it can be quasi-elliptic, see [13]).

The following facts were also observed in [12]. By the Riemann-Roch Theorem for surfaces, any irreducible curve D on X with negative square has $D^2 = -2$, and it is then rational non-singular, hence \mathbb{P}^1 . It follows that the *nef* cone $NEF(X) \subset V^+(X) \subset S_X \otimes \mathbb{R}$ (or $\mathcal{M}(X) = NEF(X)/\mathbb{R}^+ \subset \mathcal{L}(S_X) = V^+(X)/\mathbb{R}^+$) is a fundamental chamber for the reflection group $W^{(2)}(S_X) \subset O(S_X)$ generated by 2-reflections $s_\delta : x \rightarrow x + (x \cdot \delta)\delta$ in elements $\delta \in S_X$ with $\delta^2 = -2$. Moreover, classes of non-singular rational curves on X are in one to one correspondence to elements $\delta \in S_X$ with $\delta^2 = -2$ which are perpendicular to codimension one faces of $\mathcal{M}(X)$ and directed outwards. See [13, Sec. 3]. We denote this set by $P(\mathcal{M}(X))$. Here \mathbb{R}^+ is the set of all positive real numbers, $V^+(X)$ is a half-cone of the cone $V(S_X)$ of elements of $S_X \otimes \mathbb{R}$ with positive square, and $\mathcal{L}(S_X)$ is the hyperbolic space related to S_X or X . We denote by

$$A(\mathcal{M}(X)) = \{\phi \in O(S_X) \mid \phi(V^+(X)) = V^+(X) \text{ and } \phi(\mathcal{M}(X)) = \mathcal{M}(X)\}$$

the symmetry group of $\mathcal{M}(X)$, and then $\{\pm 1\}W^{(2)}(S_X) \rtimes A(\mathcal{M}(X)) = O(S_X)$ is the semi-direct product. By the theory of arithmetic groups (or integral quadratic forms theory), then the fundamental domain for $O(S_X)$ is the same as the fundamental domain for $A(\mathcal{M}(X))$ in $\mathcal{M}(X)$. In particular, this fundamental domain is a finite rational polyhedron.

It follows that *there exists only a finite number of elliptic pencils on X up to the action of $A(\mathcal{M}(X))$* . Similarly, *there exists only a finite number of non-singular rational curves on X up to the action of $A(\mathcal{M}(X))$* . Moreover, for any isotropic element $c' \in S_X$, there exists $w \in W^{(2)}(S_X)$ such that $\pm w(c')$ is *nef*. Thus *X has an elliptic fibration if and only if the Picard lattice S_X represents zero: there exists $0 \neq x \in S_X$ with $x^2 = 0$* . In particular, this is valid if $\rho(X) \geq 5$.

The fundamental result of [12] which follows from the Global Torelli Theorem for K3 surfaces (also proved in [12]) is that the action of $\text{Aut } X$ in S_X has only finite kernel (see also [13] if char $k > 0$), and for char $k = 0$ it gives a finite index subgroup in $A(\mathcal{M}(X))$. In particular, for char $k = 0$, up to

finite groups, we have natural isomorphisms of groups:

$$\mathrm{Aut} X \approx A(\mathcal{M}(X)) \cong O^+(S_X)/W^{(2)}(S_X)$$

where $O^+(S_X) = \{\phi \in O(S_X) \mid \phi(V^+(X)) = V^+(X)\}$ is the subgroup of $O(S_X)$ of index 2. It follows that *for char $k = 0$, a K3 surface X has only finite number of elliptic fibrations and non-singular rational curves up to the action of the automorphism group $\mathrm{Aut} X$.* This is the same as for all *nef* elements $h \in S_X$ with a fixed positive square $h^2 > 0$.

3 Existence of elliptic fibrations with infinite automorphism groups on K3 surfaces

Further, X is a K3 surface over an algebraically closed field k .

Let $c \in S_X$ be a primitive isotropic *nef* element. By the theory of elliptic surfaces, see e.g. [14, Ch. VII] (or by Global Torelli Theorem for K3 surfaces, if char $k = 0$), the group $\mathrm{Aut}(c)$ of automorphisms of the elliptic fibration $|c| : X \rightarrow \mathbb{P}^1$ is, up to finite index, the abelian group $\mathbb{Z}^{r(c)}$ where

$$r(c) = \mathrm{rk} c^\perp - \mathrm{rk}(c^\perp)^{(2)}. \quad (1)$$

Here c^\perp is the orthogonal complement to c in S_X (obviously, $\mathrm{rk} c^\perp = \rho(X) - 1$), and the sublattice $(c^\perp)^{(2)} \subset c^\perp$ is generated by c and by all elements with square (-2) in c^\perp . Equivalently, $(c^\perp)^{(2)}$ is generated by all irreducible components of fibres of $|c| : X \rightarrow \mathbb{P}^1$. In particular, $\mathrm{Aut}(c)$ is finite if and only if either $\rho(X) = 2$, or c^\perp is generated by c and by elements with square (-2) , up to finite index. Up to finite index, $\mathrm{Aut}(c)$ is the same as the Mordell-Weil group of the elliptic fibration c when we consider only automorphisms from $\mathrm{Aut}(c)$ which act trivially on the base \mathbb{P}^1 .

The interesting question is:

When does X have elliptic fibrations with infinite automorphism groups?

It is important, for example, for studying the dynamics of $\mathrm{Aut} X$ (e.g. see [1]) and the arithmetic of X .

The main obstruction for the existence of the fibrations in question is the finiteness of the automorphism group $\mathrm{Aut} X$ of X . Indeed, if $\mathrm{Aut} X$ is finite, then automorphism groups of all elliptic fibrations c on X are also finite since $\mathrm{Aut}(c) \subset \mathrm{Aut} X$.

Surprisingly, for $\rho(X) \geq 6$, this obstruction is sufficient and necessary according to [5], and this is valid for k of any characteristic. These results can be formulated for arbitrary hyperbolic lattices S if one fixes a fundamental chamber $\mathcal{M} \subset \mathcal{L}(S)$ for $W^{(2)}(S)$ and considers fundamental primitive isotropic elements $c \in S$ that is $\mathbb{R}^+c \in \overline{\mathcal{M}}$. Instead of $\text{Aut } X$ one should consider the symmetry group $A(\mathcal{M}) \subset O^+(S)$ or $O^+(S)/W^{(2)}(S)$.

By (1), all elliptic fibrations on X have finite automorphism groups if and only if the hyperbolic lattice $S = S_X$ satisfies the property:

$$\text{rk}(c^\perp) = \text{rk}(c^\perp)^{(2)} \quad \text{for any isotropic } c \in S. \quad (2)$$

We have the following results from [5].

Theorem 1. *Let S be an even hyperbolic lattice of rank $\rho = \text{rk } S \geq 6$ (respectively, X is a K3 surface over an algebraically closed field, and $\rho(X) \geq 6$). Then the following conditions (a), (b), (c) below are equivalent:*

(a) *S satisfies condition (2) (respectively, automorphism groups of all elliptic fibrations on X are finite).*

(b) *The group $A(\mathcal{M}) \cong O^+(S)/W^{(2)}(S)$ is finite, (respectively, $\text{Aut } X$ is finite).*

(c) *The lattice S belongs to the finite list of even hyperbolic lattices below found in [5] (respectively, $S = S_X$ is one of the lattices from the list).*

The list of lattices found in [5] is the following (we use notations from [4] and [5], which are now standard; \oplus is orthogonal sum of lattices, U is the even unimodular lattice of signature $(1, 1)$, A_n , D_m and E_k are negative definite root lattices corresponding to root systems \mathbb{A}_n , \mathbb{D}_m and \mathbb{E}_k respectively, $S(\lambda)$ is obtained from a lattice S by multiplication of its form by $\lambda \in \mathbb{Z}$, $\langle A \rangle$ is a lattice with the matrix A in some basis):

The list of all even hyperbolic lattices S with $[O(S) : W^{(2)}(S)] < \infty$ and $\text{rk } S \geq 6$ (see [5]):

$S = U \oplus 2E_8 \oplus A_1; U \oplus 2E_8; U \oplus E_8 \oplus E_7; U \oplus E_8 \oplus D_6; U \oplus E_8 \oplus D_4 \oplus A_1;$
 $U \oplus E_8 \oplus D_4, U \oplus D_8 \oplus D_4, U \oplus E_8 \oplus 4A_1; U \oplus E_8 \oplus 3A_1, U \oplus D_8 \oplus 3A_1,$
 $U \oplus A_3 \oplus E_8; U \oplus E_8 \oplus 2A_1, U \oplus D_8 \oplus 2A_1, U \oplus D_4 \oplus D_4 \oplus 2A_1, U \oplus A_2 \oplus E_8;$
 $U \oplus E_8 \oplus A_1, U \oplus D_8 \oplus A_1, U \oplus D_4 \oplus D_4 \oplus A_1, U \oplus D_4 \oplus 5A_1; U \oplus E_8, U \oplus D_8,$
 $U \oplus E_7 \oplus A_1, U \oplus D_4 \oplus D_4, U \oplus D_6 \oplus 2A_1, U(2) \oplus D_4 \oplus D_4, U \oplus D_4 \oplus 4A_1,$
 $U \oplus 8A_1, U \oplus A_2 \oplus E_6; U \oplus E_7, U \oplus D_6 \oplus A_1, U \oplus D_4 \oplus 3A_1, U \oplus 7A_1,$
 $U(2) \oplus 7A_1, U \oplus A_7, U \oplus A_3 \oplus D_4, U \oplus A_2 \oplus D_5, U \oplus D_7, U \oplus A_1 \oplus E_6;$

$U \oplus D_6, U \oplus D_4 \oplus 2A_1, U \oplus 6A_1, U(2) \oplus 6A_1, U \oplus 3A_2, U \oplus 2A_3, U \oplus A_2 \oplus A_4,$
 $U \oplus A_1 \oplus A_5, U \oplus A_6, U \oplus A_2 \oplus D_4, U \oplus A_1 \oplus D_5, U \oplus E_6; U \oplus D_4 \oplus A_1,$
 $U \oplus 5A_1, U(2) \oplus 5A_1, U \oplus A_1 \oplus 2A_2, U \oplus 2A_1 \oplus A_3, U \oplus A_2 \oplus A_3, U \oplus A_1 \oplus A_4,$
 $U \oplus A_5, U \oplus D_5; U \oplus D_4, U(2) \oplus D_4, U \oplus 4A_1, U(2) \oplus 4A_1, U \oplus 2A_1 \oplus A_2,$
 $U \oplus 2A_2, U \oplus A_1 \oplus A_3, U \oplus A_4, U(4) \oplus D_4, U(3) \oplus 2A_2.$

Thus, a K3 surface X over an algebraically closed field and with $\rho(X) \geq 6$ has an elliptic fibration with infinite automorphism group if and only if its Picard lattice S_X is different from each lattice of this finite list. If the Picard lattice S_X of X is one of lattices from the list, then not only automorphism groups of all elliptic fibrations on X are finite, but the full automorphism group $\text{Aut } X$ is finite.

If $\text{rk } S = 5$, then similar theorem is valid if one excludes two infinite series of even hyperbolic lattices, see [5].

Theorem 2. *Let S be an even hyperbolic lattice of the rank $\text{rk } S = 5$ and S is different from lattices $\langle 2^m \rangle \oplus D_4$, $m \geq 5$, and $\langle 2 \cdot 3^{2n-1} \rangle \oplus 2A_2$, $n \geq 2$ (respectively, a K3 surface X over an algebraically closed field has $\rho(X) = 5$ and S_X is different from the lattices of these two series).*

Then the following conditions (a), (b), (c) below are equivalent:

(a) *S satisfies the condition (2) (respectively, automorphism groups of all elliptic fibrations on X are finite).*

(b) *The group $A(\mathcal{M}) \cong O^+(S)/W^{(2)}(S)$ is finite, (respectively, $\text{Aut } X$ is finite).*

(c) *The lattice S belongs to the finite list of even hyperbolic lattices of rank 5 below found in [5] (respectively, S_X is one of the lattices from this list).*

If S is one of lattices $\langle 2^m \rangle \oplus D_4$, $m \geq 5$, and $\langle 2 \cdot 3^{2n-1} \rangle \oplus 2A_2$, $n \geq 2$, then S satisfies (2), but the group $A(\mathcal{M}) \cong O^+(S)/W^{(2)}(S)$ is infinite (equivalently, if S_X is one of lattices from these two series, then all elliptic fibrations on X have finite automorphism groups, but $\text{Aut } X$ is infinite if $\text{char } k = 0$).

The list of lattices of rank 5 found in [5] is as follows:

The list of all even hyperbolic lattices S with $[O(S) : W^{(2)}(S)] < \infty$ and $\text{rk } S = 5$ (see [5]):

$S = U \oplus 3A_1, U(2) \oplus 3A_1, U \oplus A_1 \oplus A_2, U \oplus A_3, U(4) \oplus 3A_1, \langle 2^k \rangle \oplus D_4,$
 $k = 2, 3, 4, \langle 6 \rangle \oplus 2A_2.$

Thus, a K3 surface X with $\rho(X) = 5$ and any char k has elliptic fibrations with infinite automorphism groups if and only if its Picard lattice S_X is different from each lattice of this finite list and from the lattices of 2 infinite series in Theorem 2. If the Picard lattice of X is one of lattices from the finite list, then not only automorphism groups of all elliptic fibrations on X are finite, but the full automorphism group $\text{Aut } X$ is finite. If the Picard lattice of X is one of lattices from the two infinite series of lattices of Theorem 2, then the automorphism groups of all elliptic fibrations on X are finite, but $\text{Aut } X$ is infinite if $\text{char } k = 0$ (if $\text{char } k > 0$, it is not known).

If the Picard number $\rho(X) = 4$ or 3 , no results, similar to that in Theorems 1 and 2, are known, except results which we shall cite below at the end of this section.

If $\rho(X) = 2$, then the automorphism groups of all elliptic fibrations on X are evidently finite. If $\rho(X) = 1$, then X has no elliptic fibrations.

In particular, Theorems 1 and 2 describe all even hyperbolic lattices S having finite group $A(\mathcal{M}) \cong O^+(S)/W^{(2)}(S)$ (they are called elliptically 2-reflective) of rank $\rho = \text{rk } S \geq 5$. Similar finite description of elliptically 2-reflective even hyperbolic lattices was obtained for $\rho = 4$ (14 lattices) in [19] (see also [9]), and for $\rho = 3$ (26 lattices) in [8].¹ Finiteness was also generalized to arbitrary arithmetic hyperbolic reflection groups and corresponding reflective hyperbolic lattices over rings of integers of totally real algebraic number fields. See [6], [7], [9] and [17] (see also [18]).

4 Elliptic fibrations with infinite automorphism groups and exceptional elements in Picard lattices for K3 surfaces

Below, X is a K3 surface over an algebraically closed field.

We consider the following general notion. For a hyperbolic lattice S and a subgroup $G \subset O(S)$, we call $x \in S$ exceptional with respect to G if its stabilizer subgroup G_x has finite index in G ; equivalently, the orbit $G(x)$ is finite. All exceptional elements with respect to G define exceptional

¹We must correct the list of lattices in [8]: the lattices $S'_{6,1,2}$ and $S_{6,1,1}$ are isomorphic.

sublattice $E \subset S$ with respect to G . Since S is hyperbolic, logically the following 4 cases are possible:

- (i) *Elliptic type of G* . The exceptional sublattice E for G is hyperbolic. Obviously, then G is finite and $E = S$. Then E is called *hyperbolic*.
- (ii) *Parabolic type of G* . The exceptional sublattice E for G is semi-negative definite and has 1-dimensional kernel. Then E is called *parabolic*.
- (iii) *Hyperbolic type of G* . The exceptional sublattice E for G is negative definite. Then E is called *elliptic*.
- (iv) *General hyperbolic type of G* . The exceptional sublattice E for G is zero.

Replacing G by the action of $\text{Aut } X$ in S_X , we obtain the following main definition. An element of the Picard lattice $x \in S_X$ is called *exceptional (with respect to $\text{Aut } X$)* if its stabilizer subgroup $(\text{Aut } X)_x$ has finite index in $\text{Aut } X$, equivalently, the orbit $(\text{Aut } X)(x)$ of x is finite.

All exceptional elements of S_X define a primitive sublattice $E(S_X)$. We call it *the exceptional sublattice of the Picard lattice (for $\text{Aut } X$)*. This sublattice was introduced in [5] (it was denoted as $R(S_X)$ in [5]), and the results which we discuss below were mentioned and in fact proved in [5] and [10], [11] (see [11, Sect. 3]). Below we just give more details.

Let us assume that X has at least one elliptic fibration c with infinite automorphism group. Then we have the following statement where for a sublattice $F \subset S_X$ we denote by $F_{pr} \subset S_X$ the primitive sublattice $F_{pr} = S_X \cap (F \otimes \mathbb{Q}) \subset S_X \otimes \mathbb{Q}$ generated by F .

Theorem 3. *Let X be a K3 surface over an algebraically closed field which has at least one elliptic fibration with infinite automorphism group.*

Then the exceptional sublattice $E(S_X)$ is equal to

$$E(S_X) = \bigcap_c (c^\perp)^{(2)}_{pr} \quad (3)$$

where c runs through all elliptic fibrations on X with infinite automorphism groups (or the Mordell–Weil groups).

In particular, two exceptional sublattices of S_X , for $\text{Aut } X$, and for the subgroup of $\text{Aut } X$ generated by Mordell–Weil groups of all elliptic fibrations with infinite automorphism groups on X , coincide.

Proof. Simple calculations, using theory of elliptic surfaces (see [14, Ch. VII]), show (see [5]) that exceptional elements for $(\text{Aut } X)_c$ (equivalently,

for the Mordell–Weil group of the elliptic fibration $|c|$ in S_X define the sublattice $(c^\perp)_{pr}^{(2)}$. It follows that $E(S_X) \subset Ell(S_X)$ where $Ell(S_X)$ is the right hand side of (3).

Since X has at least one elliptic fibration with infinite automorphism group and S_X is hyperbolic, $Ell(S_X)$ is either semi-negative definite with one dimensional kernel $\mathbb{Z}c$ (that is $Ell(S_X)$ is parabolic) when X has only one elliptic fibration c with infinite automorphism group, or $Ell(S_X)$ is negative definite (that is $Ell(S_X)$ is elliptic) if X has more than one elliptic fibrations with infinite automorphism groups. In both cases, $\text{Aut } X$ gives finite action on $Ell(S_X)$. It follows that $Ell(S_X) \subset E(S_X)$. Thus, $E(S_X) = Ell(S_X)$.

This finishes the proof. \square

Like above, for an abstract hyperbolic lattice S (replacing S_X), a fundamental chamber $\mathcal{M} \subset \mathcal{L}(S)$ for the reflection group $W^{(2)}(S)$ (replacing $\mathcal{M}(X) = NEF(X)/\mathbb{R}^+ \subset \mathcal{L}(S_X)$), and for the group $A(\mathcal{M})$ of symmetries of \mathcal{M} (replacing $\text{Aut } X$), we can similarly consider exceptional elements $x \in S$ for $A(\mathcal{M})$ and the sublattice $E(S) \subset S$ of all exceptional elements for $A(\mathcal{M})$. For a fundamental primitive isotropic element $c \in S$ for \mathcal{M} (replacing an elliptic fibration of X), we can similarly consider the stabilizer subgroup $A(\mathcal{M})_c \subset A(\mathcal{M})$ (replacing the automorphism group $\text{Aut}(c)$ of the elliptic fibration c on X). Like for K3 surfaces, we have isomorphism up to finite groups

$$A(\mathcal{M})_c \approx \mathbb{Z}^{r(c)}, \quad r(c) = \text{rk } c^\perp - \text{rk}(c^\perp)^{(2)}. \quad (4)$$

Using (4), exactly the same considerations as for Theorem 3 give similar result for arbitrary hyperbolic lattices.

Theorem 4. *Let S be a hyperbolic lattice over \mathbb{Z} , $\mathcal{M} \subset \mathcal{L}(S)$ a fundamental chamber for $W^{(2)}(S)$ and $A(\mathcal{M}) \subset O^+(S)$ its symmetry group. Let us assume that S has at least one fundamental primitive isotropic element c with infinite stabilizer subgroup $A(\mathcal{M})_c$.*

Then the exceptional sublattice $E(S)$ for $A(\mathcal{M})$ is equal to

$$E(S) = \bigcap_c (c^\perp)^{(2)}_{pr} \quad (5)$$

where c runs through all fundamental primitive isotropic elements c for \mathcal{M} with infinite stabilizer subgroups $A(\mathcal{M})_c$.

For a K3 surface X and S_X , we take $\mathcal{M} = \mathcal{M}(X) = \text{NEF}(X)/\mathbb{R}^+$. By [12], fundamental primitive isotropic elements for \mathcal{M} and elliptic fibrations on X give the same set. Right hand sides of (3) and (5) give the same. Thus, we obtain the following result which shows that calculations of exceptional sublattices of S_X for the geometric group $\text{Aut } X$ and for the lattice-theoretic group $A(\mathcal{M}(X))$ give the same.

Theorem 5. *Let X be a K3 surface over an algebraically closed field, having at least one elliptic fibration with infinite automorphism group.*

Then exceptional sublattices $E(S_X) \subset S_X$ for $\text{Aut } X$ and for $A(\mathcal{M}(X))$ are equal.

We have the following general result obtained in [5], [8], [19], [10] and [11].

Theorem 6. *For each fixed $\rho \geq 3$, the number of hyperbolic lattices S of rank ρ having non-zero exceptional sublattices $E(S) \subset S$ for $A(\mathcal{M})$ is finite.*

Proof. For hyperbolic $E(S)$ (equivalently, when $A(\mathcal{M})$ is finite), it was proved in [5], [8] and [19] (we discussed this in Sec. 3). The full list of such hyperbolic lattices S is known.

For parabolic $E(S)$, finiteness was proved in [10], but the list of such hyperbolic lattices S is not known.

For elliptic $E(S) \neq \{0\}$, it was proved in [11], but the list of such hyperbolic lattices S is not known. \square

Since $\text{rk } S_X \leq 22$ for K3 surfaces, using Theorem 6, we can introduce the following finite set of even hyperbolic lattices S of $3 \leq \text{rk } S \leq 22$:

Definition 7. *SEK3 is the set of all even hyperbolic lattices S such that $\text{rk } S \geq 3$, $E(S) \neq 0$ for $A(\mathcal{M})$ where $\mathcal{M} \subset \mathcal{L}(S)$ is a fundamental chamber for $W^{(2)}(S)$, and S is isomorphic to the Picard lattice of some K3 surface X over an algebraically closed field. By Theorem 6, the set SEK3 is finite.*

We denote by SEK3_e , SEK3_p , and SEK3_h subsets of SEK3 corresponding to $A(\mathcal{M})$ of elliptic type (i.e. finite), parabolic type, and hyperbolic type (equivalently, $E(S) = S$, $E(S)$ is semi-negative definite and has 1-dimensional kernel, $E(S) \neq 0$ is negative definite) respectively.

Combining Theorems 3 — 6, we obtain the main results.

Theorem 8. *Let X be a K3 surface over an algebraically closed field, $\rho(X) \geq 3$ and X has an elliptic fibration with infinite automorphism group. Assume that S_X is different from lattices from the finite set $\mathcal{SEK}3$.*

Then the exceptional sublattice $E(S_X) \subset S_X$ for $\text{Aut } X$ is equal to zero.

Moreover, the exceptional sublattice $E(S_X) \subset S_X$ is equal to zero for the subgroup of $\text{Aut } X$ generated by automorphism groups of all elliptic fibrations on X with infinite automorphism groups (or by their Mordell–Weil groups).

Moreover, we have the equality

$$\bigcap_c (c^\perp)^{(2)}_{pr} = E(S_X) = \{0\}. \quad (6)$$

where c runs through all elliptic fibrations on X with infinite automorphism groups.

This theorem shows that except finite number of Picard lattices from $\mathcal{SEK}3$, a K3 surface X has many elliptic fibrations with infinite automorphism groups if it has one of them: (6) gives the exact statement, “how many”. We shall also discuss directly the number of elliptic fibrations in the next section.

It would be interesting to find the finite set of Picard lattices $\mathcal{SEK}3$ of K3 surfaces. Only its subset $\mathcal{SEK}3_e$ is known.

For $\rho(X) = 1$, the exceptional sublattice is equal to S_X . For $\rho(X) = 2$, the exceptional sublattice is equal to S_X if X has an elliptic fibration. Indeed, in both these cases, $\text{Aut } X$ is finite since $O(S_X)$ is finite (this was observed in [12]). Thus, only the case of $\rho(X) \geq 3$ which we considered above is interesting.

5 Number of elliptic fibrations and elliptic fibrations with infinite automorphism groups on K3 surfaces

Using Theorem 8, we obtain the following results which show that for $\rho(X) \geq 3$, K3 surface X has infinite number of elliptic fibrations and infinite number of elliptic fibrations with infinite automorphism groups if it has one of them, if S_X is different from a finite number of exceptional Picard lattices.

Theorem 9. *Let X be a K3 surface over an algebraically closed field, $\rho(X) \geq 3$ and X has at least one elliptic fibration.*

Then X has infinite number of elliptic fibrations if S_X is different from the following finite set of Picard lattices S_X when the number of elliptic fibrations is finite:

$S_X \in \mathcal{SEK}3_e$. In particular, $\text{Aut } X$ is finite.

$S_X \in \mathcal{SEK}3_p$ and X has only one elliptic fibration. In particular, X has one elliptic fibration with infinite automorphism group, and no other elliptic fibrations.

Proof. Let us assume that X has finite number of elliptic fibrations. Then $\mathcal{M}(X)$ has only finite number of fundamental primitive isotropic elements which are all exceptional for $A(\mathcal{M}(X))$. It follows that $E(S_X) \neq 0$ and $S_X \in \mathcal{SEK}3$. Let us consider two cases.

Case 1: Let us assume that all elliptic fibrations on X have finite automorphism groups (equivalently, fundamental primitive isotropic elements c for $\mathcal{M}(X)$ have finite stabilizer subgroups $A(\mathcal{M}(X))_c$). Then $A(\mathcal{M}(X))$ is finite and $S_X \in \mathcal{SEK}3_e$. Vice a versa, if $S_X \in \mathcal{SEK}3_e$ then $\mathcal{M}(X)$ is a fundamental chamber for the arithmetic group $W^{(2)}(S_X)$ in $\mathcal{L}(S_X)$, and it has only finite number of fundamental primitive isotropic elements. Thus, X has only finite number of elliptic fibrations.

Case 2: Let us assume that X has an elliptic fibration c with infinite automorphism group $\text{Aut}(c)$. Since the number of elliptic fibrations on X is finite, all of them are exceptional for $\text{Aut } X$, and $E(S_X)$ is not zero. Since $\text{Aut } X$ is infinite, $E(S_X)$ cannot be hyperbolic. Since elliptic fibrations give isotropic elements, then $E(S_X)$ cannot be elliptic (i.e., negative definite). Thus, $E(S_X)$ is parabolic and $S_X \in \mathcal{SEK}3_p$. By Theorem 3, we have

$$E(S_X) = \bigcap_c (c^\perp)^{(2)}_{pr}$$

where c runs through all elliptic fibrations on X with infinite automorphism groups. Since $E(S_X)$ is parabolic, it follows that X has only one elliptic fibration c with infinite automorphism group and $\text{Aut } X = \text{Aut}(c)$. Since all elliptic fibrations on X are exceptional for $\text{Aut } X$, they also must have infinite automorphism groups, and they must be equal to c . Thus, X has only one elliptic fibration c .

Vice versa, let us assume that $S_X \in \mathcal{SEK}3_p$ and X has only one elliptic fibration. Then $E(S_X) \neq S_X$ and $\text{Aut } X$ is infinite. Since X has only one

elliptic fibration c , then $\text{Aut } X = \text{Aut}(c)$ is its automorphism group which is infinite.

This finishes the proof. \square

Theorem 10. *Let X be a K3 surface over an algebraically closed field, $\rho(X) \geq 3$ and X has at least one elliptic fibration with infinite automorphism group.*

Then X has infinite number of elliptic fibrations with infinite automorphism groups if S_X is different from the following finite set of Picard lattices S_X when the number of elliptic fibrations on X with infinite automorphism groups is finite:

$S_X \in \mathcal{SEK3}_p$. In particular, X has only one elliptic fibration with infinite automorphism group.

Proof. Since the number of elliptic fibrations on X with infinite automorphism groups is finite, all of them are exceptional for $\text{Aut } X$, and $E(S_X)$ is not trivial. Then $S_X \in \mathcal{SEK3}$ which is finite by Theorem 8. Since each of these elliptic fibrations is exceptional for $\text{Aut } X$ and corresponds to an isotropic element, then $E(S_X)$ cannot be elliptic (that is negative definite). Since $\text{Aut } X$ is infinite, $E(S_X)$ cannot be hyperbolic either. Thus, $E(S_X)$ is parabolic and has 1-dimensional kernel. By Theorem 3,

$$E(S_X) = \bigcap_c (c^\perp)^{(2)}_{pr}$$

where c runs through all elliptic fibrations on X with infinite automorphism groups. Since $E(S_X)$ is parabolic, it follows that X has only one elliptic fibration c with infinite automorphism group.

Vice a versa, if X has only one elliptic fibration c with infinite automorphism group, then $E(S_X) = (c^\perp)^{(2)}_{pr}$ is parabolic and $S_X \in \mathcal{SEK3}_p$.

It follows the statement. \square

If $\rho(X) = 1$ or 2 , then X has less or equal to two elliptic fibrations, and these cases are trivial.

6 Applications to K3 surfaces with exotic structures

Here we want to give some other applications of finiteness of the set of Picard lattices of K3 surfaces with non-trivial exceptional sublattice $E(S_X)$, and

elliptic fibrations with infinite automorphism group.

6.1 K3 surfaces with finite number of non-singular rational curves

Recently, D. Matsushita asked me what we can say about K3 surfaces with finite number of non-singular rational (equivalently, irreducible (-2) -curves). We have the following

Theorem 11. *A K3 surface X over an algebraically closed field has no non-singular rational curves if and only if its Picard lattice S_X has no elements with square (-2) .*

If a K3 surface X over an algebraically closed field has non-singular rational curves (equivalently, its Picard lattice S_X has elements with square (-2)), then their number is finite in the following and only the following cases (1) and (2):

- (1) $\rho(X) = 2$;
- (2) $\rho(X) \geq 3$ and the Picard lattice S_X is elliptically 2-reflective: $[O(S_X) : W^{(2)}(S_X)] < \infty$. The number of elliptically 2-reflective hyperbolic lattices is finite, and they are enumerated in [5], [8] and [19] (for $\rho(X) \geq 5$ see their lists in Sec. 3).

Proof. Let $S = S_X$ and $\mathcal{M} = \mathcal{M}(X)$ be the fundamental chamber for $W^{(2)}(S)$. Then the non-singular rational curves on X are in one to one correspondence to elements of the set $P(\mathcal{M})$ of perpendicular vectors to \mathcal{M} with square (-2) and directed outwards.

If S has no elements with square (-2) , then $P(\mathcal{M})$ is empty and X has no non-singular rational curves.

Let us assume that S has an element δ with $\delta^2 = -2$. Then $\pm w(\delta)$ gives one of the elements of $P(\mathcal{M})$ for some $w \in W^{(2)}(S)$, the set $P(\mathcal{M})$ is not empty, and X contains a non-singular rational curve.

Since S is hyperbolic and S has elements with square (-2) , then $\rho(X) = \text{rk } S \geq 2$.

Let $\rho(X) = 2$. Then $\mathcal{M} = V^+(S)/\mathbb{R}^+$ is an interval, and elements of $P(\mathcal{M})$ correspond to terminals of this interval. Thus, $P(\mathcal{M})$ has not more than 2 elements, and the number of non-singular rational curves on X is one or two.

Let $\rho(X) \geq 3$, and $P(\mathcal{M})$ is non-empty and finite. Then all elements of $P(\mathcal{M})$ are exceptional for the symmetry group $A(\mathcal{M}) \subset O^+(S)$ of \mathcal{M} , and

the exceptional sublattice $E(S)$ is not zero. Then by Theorem 6 (from [5], [8], [19], [10] and [11]), S is one of a finite number of hyperbolic lattices of rank ≤ 22 .

Actually, the main idea of the proof of this theorem in [8], [10] and [11] is that $P(\mathcal{M})$ has elements $\delta_1, \dots, \delta_\rho \in S$, $\rho = \text{rk } S$, which generate $S \otimes \mathbb{Q}$ and $\delta_i \cdot \delta_j \leq 19$, $1 \leq i < j \leq \rho$. (These elements define a narrow part of \mathcal{M} .) It follows that $P(\mathcal{M})$ generates $S \otimes \mathbb{Q}$, the group $A(\mathcal{M}) \cong O^+(S)/W^{(2)}(S)$ is finite since $P(\mathcal{M})$ is finite, the lattice S is elliptically 2-reflective: $[O(S) : W^{(2)}(S)] < \infty$, and the number of such lattices is finite.

This finishes the proof. \square

6.2 K3 surfaces with finite number of Enriques involutions

Here we restrict to basic fields k of char $k \neq 2$.

We recall that an involution σ on a K3 surface X over an algebraically closed field k of char $k \neq 2$ is called *Enriques involution* if σ has no fixed points on X . Then $X/\{\text{id}, \sigma\}$ is Enriques surface. See [2]. It is well-known (see [2]) that σ in S_X has the eigen-value 1 part which is isomorphic to the standard hyperbolic lattice $S_X^\sigma \cong U(2) \oplus E_8(2)$ of rank 10. A general K3 surface with Enriques involution has $S_X = S_X^\sigma \cong U(2) \oplus E_8(2)$, and only finite number of Enriques involutions (if char $k = 0$, it is unique).

We have the following result.

Theorem 12. *Let X be a K3 surface over an algebraically closed field k of char $k \neq 2$ and X has an Enriques involution.*

If X has only finite number of Enriques involutions, then either S_X is isomorphic to $U(2) \oplus E_8(2)$, or S_X belongs to the finite set $SEK3$.

In particular, if S_X is different from lattices of these two finite sets, then X has infinite number of Enriques involutions.

Proof. Let σ be an Enriques involution on X .

Since $S_X^\sigma \cong U(2) \oplus E_8(2)$ is a sublattice of S_X , it follows that $\rho(X) \geq 10$.

Let $\rho(X) = 10$. Then $S_X = S_X^\sigma$, and σ is the identity on S_X . Since $\text{Aut } X$ has only finite kernel in S_X , it follows that X has only finite number of Enriques involutions (it is unique if char $k = 0$).

Let $\rho(X) > 10$. Then for each Enriques involution σ on X , the orthogonal complement $S_\sigma = (S_X^\sigma)^\perp$ in S_X is a non-zero negative definite sublattice of

S_X which has a finite automorphism groups. If X has only finite number of Enriques involutions, then all these orthogonal complements are contained in the exceptional sublattice $E(S_X)$ of S_X for $\text{Aut } X$, and $E(S_X) \neq \{0\}$. Since $\rho(X) \geq 10 \geq 6$, then by Theorem 1 either $\text{Aut } X$ is finite and X has only finite number of Enriques involutions, and $S_X \in \mathcal{SEK}3_e$, or X has an elliptic fibration with infinite automorphism group. By Theorem 5, then exceptional sublattices of S_X for $\text{Aut } X$ and for $A(\mathcal{M}(X))$ are the same, and $S_X \in \mathcal{SEK}3$. By Theorem 6 (from [5], [8], [19], [10] and [11]), the set $\mathcal{SEK}3$ is finite.

This proves the theorem. \square

The method of the proof is so general, that by the same considerations, one can prove similar results for other types of involutions or automorphisms on K3 surfaces, and other structures on K3 surfaces.

6.3 K3 surfaces with naturally arithmetic automorphism groups

This is related to the recent preprint by B. Totaro [15].

Definition 13. *Let X be a K3 surface over an algebraically closed field, and S_X its Picard lattice.*

We say that the automorphism group $\text{Aut } X$ is naturally arithmetic, if there exists a sublattice $K \subset S_X$ such that the action of $\text{Aut } X$ in S_X identifies $\text{Aut } X$ as a subgroup of finite index in $O(K)$. More precisely, there exists a subgroup $G \subset \text{Aut } X$ of finite index such that K is G -invariant, and the natural homomorphism $G \rightarrow O(K)$ has finite kernel and cokernel.

For example, if $\text{Aut } X$ is finite, then one can take $K = \{0\} \subset S_X$, and $\text{Aut } X$ is naturally arithmetic. Thus, all K3 surfaces with elliptically 2-reflective Picard lattices S_X (for $\rho(X) \geq 5$ see there list in Sec. 3) have naturally arithmetic automorphism groups.

We have the following result which uses the Global Torelli Theorem for K3 surfaces [12], and it is valid over \mathbb{C} (or over an algebraically closed field k of char $k = 0$).

Theorem 14. *Let X be a K3 surface over \mathbb{C} . Then $\text{Aut } X$ is naturally arithmetic in the following and only the following cases (1), (2) and (3):*

- (1) *The Picard lattice S_X has no elements with square (-2) .*

(2) S_X has elements with square (-2) and $\rho(X) = 2$.

(3) S_X has elements with square (-2) , $\rho(X) \geq 3$, and S_X is one of lattices from the subset (it will be described in the proof) of the finite set $SEK3$.

In particular, if $\rho(X) \geq 3$ and S_X has elements with square (-2) , then $\text{Aut } X$ is not naturally arithmetic, except a finite number of Picard lattices S_X .

Proof. We identify $\text{Aut } X$ with its action in S_X . By [12], the automorphism group $\text{Aut } X$ is a subgroup of finite index in $A(\mathcal{M})$ where $\mathcal{M} = \mathcal{M}(X)$, and $O^+(S_X) = A(\mathcal{M}) \ltimes W^{(2)}(S_X)$ is the semi-direct product. We can consider $A(\mathcal{M})$ instead of $\text{Aut } X$. Thus, the natural arithmeticity of $\text{Aut } X$ depends on S_X only. If S_X has no elements with square (-2) , then $W^{(2)}(S_X)$ is trivial, and $A(\mathcal{M})$ and $\text{Aut } X$ are naturally arithmetic (one can take $K = S_X$). We obtain the case (1).

If S_X has elements with square (-2) and $\rho(X) = 2$, then $A(\mathcal{M})$ and $\text{Aut } X$ are finite, and they are naturally arithmetic (one can take $K = \{0\}$). We obtain the case (2).

Let us assume that S_X has elements with square (-2) , $\rho(X) \geq 3$ and $\text{Aut } X$ is naturally arithmetic for some sublattice $K \subset S_X$. Let us show that then the exceptional sublattice $E(S_X)$ for $A(\mathcal{M})$ (or $\text{Aut } X$) is not trivial.

Let us assume that $W^{(2)}(S_X)$ is finite. The group $W^{(2)}(S_X)$ is generated by reflections s_δ where $\delta \in P(\mathcal{M})$, and all such reflections are different. It follows that $P(\mathcal{M})$ is finite and non-empty (equivalently, the number of non-singular rational curves on X is finite and non-empty). By Theorem 11, then S_X is elliptically 2-reflective: $[O(S_X) : W^{(2)}(S_X)] < \infty$, the groups $A(\mathcal{M})$ and $\text{Aut } X$ are finite, and the exceptional sublattice $E(S_X) = S_X$ is not trivial.

Let us assume that $W^{(2)}(S_X)$ is infinite. Then $W^{(2)}(S_X)$ and $O^+(S_X)$ act transitively on infinite number of fundamental chambers for $W^{(2)}(S_X)$ in $\mathcal{L}(S_X)$. But $A(\mathcal{M})$ sends \mathcal{M} to itself. Thus, $A(\mathcal{M})$ has infinite index in $O^+(S_X)$. It follows that $K \subset S_X$ has $\text{rk } K < \text{rk } S_X$, and the orthogonal complement $E = K^\perp$ in S_X is not zero.

If K is negative definite, then $A(\mathcal{M})$ and $\text{Aut } X$ are finite, and $E(S_X) = S_X$ is not trivial.

If K is semi-negative definite and not negative definite, then it has a one-dimensional kernel $\mathbb{Z}c$, where c is exceptional, and $E(S_X)$ is not trivial.

If K is hyperbolic, then $E = K^\perp$ is negative definite and non-zero. It gives a non-trivial sublattice in $E(S_X)$ since E has a finite automorphism

group. Thus, $E(S_X)$ is not trivial.

By Theorem 6, the lattice S_X is one of a finite number of hyperbolic lattices S with $3 \leq \text{rk } S \leq 22$ and with non-trivial exceptional sublattice $E(S)$ for $A(\mathcal{M})$. Thus, S_X belongs to the finite set $\mathcal{SEK3}$ of hyperbolic lattices.

This proves the theorem. \square

Because of Theorem 14, the following result is important. We know it for many years, and it is a corollary of results of [4]. As we know, it was never published.

Theorem 15. *Let X be a K3 surface over \mathbb{C} , and $\rho(X) = \text{rk } S_X \geq 12$.*

Then S_X has elements with square (-2) . In particular, X contains a non-singular rational curve \mathbb{P}^1 .

Proof. The Picard lattice $S = S_X$ is a primitive sublattice of the lattice $H^2(X, \mathbb{Z}) = L$ which is an even unimodular lattice of signature $(3, 19)$. It is unique up to isomorphisms. The transcendental lattice $T = S_L^\perp$ has rank ≤ 10 , and $\text{rk } T + \text{rk}(T^*/T) \leq 20 = \text{rk } L - 2$. By [4, Theorem 1.14.4], the lattice T has a unique primitive embedding into L , up to isomorphisms. Thus, for any primitive embedding $T \subset L$, we have $(T)_L^\perp \cong S = S_X$.

On the other hand, by [4, Theorem 1.12.2], the lattice $T \oplus \langle -2 \rangle$ has a primitive embedding into L . For this primitive embedding, T_L^\perp contains a sublattice $\langle -2 \rangle$. Thus, $S = S_X$ also contains a primitive sublattice $\langle -2 \rangle$. Equivalently, there exists $\delta \in S_X$ with $\delta^2 = -2$.

This proves the theorem. \square

From Theorems 14 and 15, we obtain

Corollary 16. *Up to isomorphisms, there exists only a finite number of Picard lattices S_X of K3 surfaces over \mathbb{C} such that $\text{rk } S_X = \rho(X) \geq 12$ and $\text{Aut } X$ is naturally arithmetic.*

In contrary, by [4, Theorem 1.12.4], any even hyperbolic lattice S of $\text{rk } S \leq 11$ has a primitive embedding into even unimodular lattice of signature $(3, 19)$. Thus, by epimorphicity of Torelli map for K3 surfaces, [3], the lattice S is isomorphic to Picard lattice S_X of a K3 surface X over \mathbb{C} . It follows that for each $1 \leq \rho \leq 11$, there exists infinite number of non-isomorphic Picard lattices S_X of K3 surfaces over \mathbb{C} such that $\text{rk } S_X = \rho$, S_X has no elements with square (-2) and then $\text{Aut } X$ is naturally arithmetic.

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